Higher-order Rogue Waves in a New (2+1)D Integrable Water Wave Model
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1. Objective
- To construct rogue wave solutions of a new integrable (2+1)-dimensional Boussinesq equation by using Bell polynomial, Hirota’s bilinearization and a generalized polynomial function.
- To explore the dynamics of these localized structures and manipulation of their identities by tuning arbitrary parameters.

2. Introduction
We consider the following (2+1)D integrable Boussinesq equation governing the gravity waves and collisions of surface water waves proposed recently:

\[ u_{tt} - u_{xx} - \beta (u^2)_{xx} - \gamma u_{xxxx} + \alpha_2 u_{yy} + \alpha u_{xx} = 0, \]  

where \( \alpha, \beta \) and \( \gamma \) are nonzero constants. Equation is found to be integrable and passes Painlevé integrability test. Equation (1) shall reduces to different versions of Boussinesq and Benjamin-Ono type models for suitable choices of \( \alpha, \beta \) and \( \gamma \). Results on solitary waves, rational solutions, periodic and lump solutions to various types of (2+1)D and (1+1)D Boussinesq equations are reported already.

3. Methodology
The Hirota bilinear method and Bell polynomial are used.

Bilinear transformation: \[ u(x, \tau) = \frac{3\gamma}{\beta} q_{xx} + u_0, \]  

where \( \tau = \alpha_3 t, \quad x = x, \quad \text{and} \quad q = 2 \ln(R(u(x, \tau))). \]

Billinear form of Eq. (1):

\[ \left( a_t^2 + \frac{\alpha_2}{4} + \alpha_3 \right) D_x^2 - (2\beta u_0 + 1) D_x^2 - \gamma D_x^2 \cdot R = 0. \]  

Utilize the following generalized ansatz:

\[ R_{r+1}(x, \tau; \lambda, \mu) = R_{r+1} + 2\lambda \tau F_r + 2\mu x G_r + (\lambda^2 + \mu^2) R_{r-1}, \]  

with

\[ R_r(x, \tau) = \sum_{n=0}^{(r+1)/2} \sum_{i=0}^{(r+1)-2n} c_{r+1-2n} x^{r+1-2n} \tau^{r+1-2n}, \]  

\[ F_r(x, \tau) = \sum_{i=0}^{(r+1)/2} \sum_{n=0}^{(r+1)-2n} c_{r+1-2n} x^{r+1-2n} \tau^{r+1-2n}, \]  

\[ G_r(x, \tau) = \sum_{i=0}^{(r+1)/2} \sum_{n=0}^{(r+1)-2n} h_{r+1-2n} x^{r+1-2n} \tau^{r+1-2n}, \]

where \( \lambda, \mu, c_{r+1}, c_{r+1}, e_{r+1}, h_{r+1} \) are arbitrary real parameters.

4. First-Order Rogue Wave
First order rogue wave for \( r = 0 \) in (4a):

\[ R = R_1(x, \tau) = c_0 x + c_0 x^2 + c_{2,0} x^2. \]  

\[ u = u_0 + 12\gamma \left( \frac{3}{(1+2\mu_0)} - \frac{(x-\lambda)^2 - (1+2\mu_0)}{(2\mu_0 + 1)} (y + \alpha t - \mu) \right)^{\frac{3}{(1+2\mu_0)}} + \frac{(x-\lambda)^2 - (1+2\mu_0)}{(2\mu_0 + 1)} (y + \alpha t - \mu). \]  

**Figure 1:** Doubly localized bright and dark rogue waves in \( x - t \) & rational soliton in \( y - t \).

**Figure 2:** Role of arbitrary parameters \( \beta, \alpha \) and \( \mu \).

* Constraint conditions: \( 2\alpha_1 + \alpha \neq 0 \) and \( 1 + 2\mu_0 \beta < 0 \).
* \( u_0 \), \( \beta \), and \( \gamma \): determine the type (bright or dark) rogue wave.
* \( \alpha_1 \) and \( \alpha \): directly proportional to the amplitude and tail-depth.
* \( \mu \): inversely proportional/affecting the amplitude and tail-depth.
* \( \alpha \): shifts the position of rogue wave along the x-axis.

5. Second-Order Rogue Wave
Second-order rogue wave for \( r = 1 \) in Eq. (4a):

\[ R_2 = R_2(x, \tau) + 2\lambda \tau F_2(x, \tau) + 2\mu x G_2(x, \tau) + (\lambda^2 + \mu^2) R_1, \]  

\[ u = u_0 + \frac{\mu}{\beta} (\ln R_2). \]  

**Figure 3:** Doubly localized rogue waves in \( x - t \) & singly-localized rational soliton in \( y - t \).

6. Third-Order Rogue Wave
Take \( r = 2 \) in (4a) to extract third-order rouge wave solution.

\[ R_3 = R_3(x, \tau) + 2\lambda \tau F_3(x, \tau) + 2\mu x G_3(x, \tau) + (\lambda^2 + \mu^2) R_2, \]

\[ u = u_0 + \frac{\mu}{\beta} (\ln R_3). \]  

**Figure 4:** Doubly localized third-order bright and dark rogue waves in \( x - t \) & singularly-localized bright and dark rational solitons in \( y - t \).

7. Discussion & Conclusions
- Considered a new integrable (2+1)-dimensional Boussinesq equation.
- Constructed higher-order rogue wave solutions by using Bell polynomial, Hirota’s bilinearization and a generalized polynomial function.
- Explored the dynamics of these bright and dark localized structures and manipulation of their identities by tuning arbitrary parameters.
- The arbitrary parameters \( (u_0, \beta, \gamma, \lambda, \alpha, \mu) \) help to control the amplitude/dearth, width, tail-depth, and localization of the bright and dark rogue waves & rational solitons.
- We have observed the evolution dynamics of W-shaped, M-shaped, and multi-peak rational solitons.
- The results presented in this work will be encouraging to the studies on the rogue waves on other higher dimensional systems.
- The study will be helpful to experimental investigations on the controlling mechanism of rogue waves in optical systems, atomic condensates, deep water oceanic waves, and other related coherent wave systems.

**Reference**

**Acknowledgments**
KS was supported by National Post-Doctoral Fellowship (File No. PDF/2016/000547) of DST–SERB, Govt. of India.